

A generalized Ginsparg-Wilson relation

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Abstract

We show that, under certain general assumptions, any sensible lattice Dirac operator satisfies a generalized form of the Ginsparg-Wilson relation (GWR). Those assumptions, on the other hand, are mostly dictated by large momentum behaviour considerations. We also show that all the desirable properties often deduced from the standard GWR hold true of the general case as well; hence one has, in fact, more freedom to modify the form of the lattice Dirac operator, without spoiling its nice properties. Our construction, a generalized Ginsparg-Wilson relation (GGWR), is satisfied by some known proposals for the lattice Dirac operator. We discuss some of these examples, and also present a derivation of the GGWR in terms of a renormalization group transformation with a blocking which is not diagonal in momentum space, but nevertheless commutes with the Dirac operator.

1 Introduction

The construction of chirally symmetric theories on the lattice has been the subject of renewed interest during the last years, because of some interesting developments. Among these, the overlap formalism [1], based on an earlier idea by Kaplan [2], has been shown to be a quite satisfactory approach to the

study of chirality-sensitive perturbative and non-perturbative phenomena on the lattice.

On the other hand, the Nielsen-Ninomiya theorem [3] tells us that a strictly chirally symmetric theory on the lattice necessarily breaks some nice features one would expect the theory to have, like proper continuum limit for the fermion propagators, absence of doublers, and locality. However, there is a compromise solution, which amounts to breaking chiral symmetry, but in such a way that most of the interesting properties are preserved. This can be achieved if the Dirac operator D satisfies the Ginsparg-Wilson relation [4]

$$D\gamma_5 + \gamma_5 D = aD\gamma_5 D . \quad (1)$$

which can be thought of as a mild deformation of the ‘strong’ anticommutation relation,

$$D_c\gamma_5 + \gamma_5 D_c = 0 , \quad (2)$$

corresponding to a chirally symmetric operator D_c .

An important consequence of (1), is that the fermionic action defined in terms of such Dirac operator ¹

$$S_F = a^4 \sum_x \bar{\psi} D \psi , \quad (3)$$

is invariant under the global symmetry transformation [5]

$$\begin{aligned} \delta\psi &= i\xi\gamma_5(1 - \frac{1}{2}aD)\psi \\ \delta\bar{\psi} &= i\xi\bar{\psi}(1 - \frac{1}{2}aD)\gamma_5 , \end{aligned} \quad (4)$$

where ξ is a (real) infinitesimal parameter. The finite- ξ version of (4) is, of course,

$$\begin{aligned} \psi &\rightarrow \exp[i\xi\gamma_5(1 - \frac{1}{2}aD)]\psi \\ \bar{\psi} &\rightarrow \bar{\psi} \exp[i\xi(1 - \frac{1}{2}aD)\gamma_5] . \end{aligned} \quad (5)$$

Thus the fermionic action has, on the lattice, a symmetry which in the naive continuum limit ($a \rightarrow 0$) coincides with the usual chiral transformation.

¹We shall follow here the conventions of [5].

Note, however, that the lattice symmetry transformation itself is quite non-local.

Besides allowing for the existence of this symmetry, it has been emphasized [6, 7] that the general solution of (1) can be written as

$$D = D_c(1 + \frac{1}{2}aD_c)^{-1} = (1 + \frac{1}{2}aD_c)^{-1}D_c, \quad (6)$$

D_c being a lattice Dirac operator satisfying (2). This implies the important property that D and D_c have the same zero modes. Indeed, it has been shown that

$$\text{index}(D) = \text{index}(D_c), \quad (7)$$

and this is the reason why (6), as a mapping from D_c to D , has been called [8] a ‘topological transformation’.

Thus the problem reduces to finding a suitable (non-local) operator D_c , satisfying the relation (2). It should also have the proper index (the topological transformation cannot change it). Inserting that operator into (6) yields D , which is no longer chirally symmetric ($\{D, \gamma_5\} \neq 0$), but which, for a suitable D_c , will be local. This could be summarized by saying that the non-locality of the chirally symmetric D_c is traded for the locality of D , at the price of introducing non-local chiral symmetry transformations.

In this paper, we shall argue that the GWR is the simplest form of the more general GGWR, and that the important properties often associated to the GWR are also valid for the generalized relation. Moreover (see section 4 below), this generalization appears quite naturally in some examples. The different realizations of the GGWR will be shown to correspond to different ways of fixing the UV behaviour of the Dirac operator for large eigenvalues. As for the case of the GWR, its generalization also requires the existence of a suitable D_c , while the relation between D and D_c will be different to (6). This introduces new parameters and different ways of adjusting the locality properties of D .

The structure of this paper is as follows: in section 2, we derive the generalization of the standard GWR, discussing the nature of the different constraints one has to impose, and their consequences. Then the general solution of the GGWR is presented, in terms of the solution of a (c-number) equation. We also discuss the meaning of the GGWR in the context of renormalization group transformations with a non-trivial averaging function.

Section 3 deals with the ‘topological’ properties of the lattice Dirac operator, such as its index theorems. Finally, in section 4, we consider some known proposals for lattice Dirac operators, showing that they satisfy a GGWR.

2 Generalizing the GWR

We begin by observing that, for the standard GWR case, the finite transformations (5) that leave the fermionic action (3) invariant are *non-local*, because they include an exponential of the (local) operator D , while they become local in the infinitesimal case. It is rather strange to have a *global* symmetry that makes such a radical distinction between finite and infinitesimal transformations. A natural, more symmetric situation, would be to consider transformations that are not necessarily local, even at the infinitesimal level:

$$\begin{aligned}\delta\psi &= i\xi\gamma_5 f(aD)\psi \\ \delta\bar\psi &= i\xi\bar\psi f(aD)\gamma_5,\end{aligned}\tag{8}$$

where f is a real function with the proper continuum limit behaviour

$$x \rightarrow 0 \Rightarrow f(x) \rightarrow 1,\tag{9}$$

a condition that must be satisfied if we want (8) to be the lattice version of the continuum chiral symmetry transformations. This is the first of a series of constraints on f ; the following ones will appear, as we shall see, by relating f to the actual form of the lattice Dirac operator. We note that relation (4) is, of course, a particular case of (8), i.e., $f(x) = 1 - \frac{1}{2}x$. Before proceeding, we remark that everything we discuss in this section can be applied also to a theory in the continuum, if the basic assumptions are true for the system under consideration. Of course, one must then replace a by Λ^{-1} , where Λ is the UV cutoff.

Of course not every D will make (3) invariant under (8). To see this, we *impose* (8) to be a symmetry of (3), and find that this requires D to satisfy

$$D\gamma_5 f(aD) + f(aD)\gamma_5 D = 0,\tag{10}$$

which is the GGWR ².

²It is trivial to check that it yields the usual GWR if f is assumed to be linear.

To begin with our study of (10), we first look for its solutions, a problem which, except for the particular case of a linear f , would seem to involve solving a non-linear operatorial equation. This is fortunately not so, as we shall now see. We first left and right multiply both sides of (10) by $\frac{1}{f(aD)}$, obtaining

$$\frac{1}{f(aD)} D \gamma_5 + \gamma_5 \frac{1}{f(aD)} D = 0 , \quad (11)$$

where we used the fact that $f(aD)$ commutes with D . It is evident from (11) that the general solution is

$$\frac{1}{f(aD)} D = D_c \quad (12)$$

with D_c as in the case of the usual GWR. Now the non-linear operatorial equation (12) yields an implicit definition for D in terms of D_c . The problem is greatly simplified by noting that from (12), and from the fact that f commutes with D , one deduces

$$D = f(aD)D_c = D_c f(aD) \quad (13)$$

namely, $f(aD)$ commutes with D_c . Hence, also D commutes with D_c ,

$$DD_c = f(aD)D_c^2 = D_c f(aD)D_c = D_c D . \quad (14)$$

As D and D_c commute, we see that the problem of solving equation (12) explicitly for D in terms of D_c reduces to that of finding the (numerical) function g such that $x = g(y)$ solves:

$$\frac{x}{f(x)} = y . \quad (15)$$

For such a solution to exist around a given point x , a necessary condition is

$$\frac{d}{dx} \frac{x}{f(x)} \neq 0 \quad (16)$$

or

$$f(x) \neq C x \quad (17)$$

with C a real constant. We may then be sure that such a g exists, at least near to $x = 0$, because of condition (9). To address different x 's, we shall

now study in more detail the relation between D and D_c . It is clear that, when such a g exists, then we can relate D and D_c by:

$$aD = g(aD_c) , \quad (18)$$

which is the generalization of (6). We note that, also because of condition (9),

$$x \sim 0 \Rightarrow g(y) \sim y \quad (19)$$

and thus $D \sim D_c$ for $aD \sim 0$. This means that both D and D_c have approximately the same low-lying modes³. This is true in particular of the zero modes (see section 3 below).

Let us now consider the large- x regime. We note that the fermionic determinant (in an A background), $\mathcal{Z}(A)$, corresponding to the action (3) can be written as

$$\mathcal{Z}(A) = \det aD(A) = \det [a\gamma_5 D(A)] = \Pi_n (a\lambda_n) \quad (20)$$

where λ_n are the eigenvalues of $\gamma_5 D$, which we order according to their increasing moduli. The a 's were introduced for the sake of the normalization, and the γ_5 for algebraic simplicity (note that D and $\gamma_5 D$ have identical determinants).

Because of (18), we may also write

$$\mathcal{Z}(A) = \Pi_n g(a\lambda_n^c) \quad (21)$$

where λ_n^c are the eigenvalues of $\gamma_5 D_c$. The large eigenvalue (large n , because of our ordering) regime is characterized by $a\lambda_n^c \gg 1$. A necessary condition for the determinant to be finite is

$$\lim_{n \rightarrow \infty} g(a\lambda_n^c) = 1 \quad (22)$$

since this is required for the convergence of the infinite product. Namely,

$$y \rightarrow \infty \Rightarrow g(y) \rightarrow 1 . \quad (23)$$

Condition (23), when combined with (15), yields another important constraint on f :

$$x \rightarrow 1 \Rightarrow f(x) \rightarrow 0 . \quad (24)$$

³To work with Hermitian operators, one can of course multiply (18) to the left by γ_5 .

The most immediate consequence of (24) is that the symmetry transformation (8) is trivialized at the large eigenvalues, namely, it becomes the identity transformation for modes of the order of the cutoff. More explicitly,

$$\gamma_5 f(aD) \phi = 0 \quad (25)$$

if ϕ is an eigenvector of $\gamma_5 D$, with an eigenvalue of the order of the cutoff.

Considering both (9) and (24), we may conclude that the generalized transformations (8) have the property of behaving as the usual chiral transformations for low eigenvalues (large distances), and disappearing for large eigenvalues (short distances). The non-local nature of the transformation stems from the fact that it interpolates between these two regimes.

As a final remark on the general solution of (10), we check that for the linear- f case (GWR), we have $f(x) = 1 - \frac{1}{2}x$, and $g(y) = \frac{y}{1+\frac{y}{2}}$, thus

$$D = D_c(1 + \frac{1}{2}aD_c)^{-1}, \quad (26)$$

as it should be. Contrary to what happens for the linear- f case, not every choice of $f(x)$ will allow us to find D in terms of D_c exactly. However, there are important non-linear cases where such a solution can indeed be found, as discussed in section 4.

It is convenient to adopt the convention that $\gamma_5 D_c$ is Hermitian. Together with (18), this implies that $\gamma_5 D$ is also Hermitian, as can be easily verified.

Let us now study the GGWR from the point of view of the symmetry that remains after performing a renormalization group transformation, which is the approach followed in the original derivation of the GWR [4]. To that end, we start by considering $\mathcal{Z}[A]$, the partition function for massless Dirac fermions in the presence of an external gauge field:

$$\mathcal{Z}[A] = \int \mathcal{D}\bar{\psi} \mathcal{D}\psi e^{-S_F[\bar{\psi}, \psi; A]} \quad (27)$$

where S_F denotes the fermionic action. Since we want to deal with both the continuum and lattice cases, we use the notation:

$$S_F[\bar{\psi}, \psi; A] = \int_{x,y} \bar{\psi}_x D_c(x, y) \psi_y \quad (28)$$

where the integral should be interpreted as a sum, when considering the lattice case. $D_c(x, y)$ is the chirally symmetric Dirac operator. Then, following [4], we introduce a blocking transformation to obtain an effective action

for the new fermion variables $\bar{\chi}, \chi$:

$$e^{-S_{eff}[\bar{\chi}, \chi; A]} = \frac{1}{\mathcal{N}} \int \mathcal{D}\bar{\psi} \mathcal{D}\psi \times e^{-S_F[\bar{\psi}, \psi; A] - \mathcal{T}[\bar{\chi}, \chi; \bar{\psi}, \psi; A]}, \quad (29)$$

where \mathcal{N} is a normalization constant and \mathcal{T} defines the blocking. The general form of this functional, for the non-interacting fermion case is

$$\mathcal{T}[\bar{\chi}, \chi; \bar{\psi}, \psi; A] = \int_{x,y} (\bar{\psi}_x - \bar{\chi}_x) \alpha(x, y) (\psi_y - \chi_y) \quad (30)$$

where alpha determines, of course, the properties of the transformation. When it is a local function, diagonal in Dirac and flavour space, the usual GWR emerges [4]. This can be done following the original derivation in the reference above, or by explicitly evaluating the functional integral over the original fields, and comparing with (6). Performing the functional integral over the fermionic fields, we obtain

$$S_{eff}[\bar{\chi}, \chi; A] = \int_{x,y} \bar{\chi}_x D(x, y) \chi_y \quad (31)$$

with

$$D = \frac{D_c}{1 + \alpha^{-1} D_c}. \quad (32)$$

The simplest choice corresponds to taking a constant α , which by dimensional reasons has to be proportional to the cutoff, $\alpha \propto \Lambda \propto 1/a$. In fact, relation follows from the choice $\alpha = \frac{2}{a}$.

We now show that the GGWR may be obtained from a blocking transformation which is non diagonal in Dirac space. We note that α has to commute with D_c , otherwise, more symmetries would be broken. As this has to happen regardless of the particular background field configuration, α can only be a function of D_c :

$$\alpha = \frac{2}{a \beta[a D_c]}, \quad (33)$$

where β is a dimensionless real function. In this case, we have of course

$$D = \frac{D_c}{1 + \frac{a}{2} \beta[a D_c] D_c}. \quad (34)$$

This allows us to consider the effect of iterating the blocking transformation. Denoting by $D^{(0)}$ the initial operator D_c , and by $D^{(n)}$ the one that results

from applying the transformation n times, we see that

$$D^{(n)} = \frac{D_c}{1 + \frac{na}{2}\beta[aD_c]D_c} . \quad (35)$$

That the operator D in (34) will satisfy a GGWR is evident from the fact that the relation between D and D_c implies, in the notation of section 2,

$$x = g(y) = \frac{y}{1 + \frac{1}{2}\beta(y)y} , \quad (36)$$

and this implies a non linear $f(x)$, except for a trivial function β .

3 Topological properties

Let us first check the property of ‘topological invariance’ of the transformation (18), namely, whether it preserves the index of D_c or not. To that end, let ϕ_0^\pm be a chiral zero mode of D_c :

$$D_c \phi_0^\pm = 0 \quad (37)$$

where \pm denotes the chirality of ϕ_0^\pm

$$(\frac{1 \pm \gamma_5}{2})\phi_0^\pm = \phi_0^\pm . \quad (38)$$

Then,

$$D\phi_0^\pm = \frac{1}{a}g(aD_c)\phi_0^\pm , \quad (39)$$

which does, indeed, yield

$$D\phi_0^\pm = 0 \quad (40)$$

We are using property (19) for the regular behaviour of $g(x)$ near to the origin. As $g(x) \sim x$ for $x \rightarrow 0$, then the inverse of g also behaves like $\sim x$ around $x = 0$. This guarantees the validity of the reciprocal proposition: a zero mode of D is also a zero mode of D_c . Thus (18) is topologically invariant, as expected from the fact that the lowest modes of D and D_c are not distorted by that transformation.

Let us now study the index theorems on the lattice. We shall follow the approach developed in [9, 10], adapting it to the GGWR. As a first check,

we verify that the global version of the index theorem, obtained from the anomalous Jacobian for a chiral transformation, holds true for the GGWR. Following [9], we have to consider the trace

$$\text{Tr} [\gamma_5 f(aD)] \quad (41)$$

which corresponds to half of the Jacobian factor for the transformation (8). Evaluating the trace in the basis of the ϕ_n , eigenvectors of the Hermitian operator $\gamma_5 D$, we see that

$$\begin{aligned} \text{Tr} [\gamma_5 f(aD)] &= \sum_n \phi_n^\dagger \gamma_5 f(aD) \phi_n \\ &= \sum_{\lambda_n=0} \phi_n^\dagger \gamma_5 \phi_n + \sum_{\lambda_n \neq 0} \phi_n^\dagger \gamma_5 f(aD) \phi_n \end{aligned} \quad (42)$$

as a consequence of (9). To deal with the sum over non-zero modes, we left multiply relation (10) by γ_5 and sandwich it with an arbitrary non-zero mode ϕ_n , obtaining

$$\lambda_n \phi_n^\dagger \gamma_5 f(aD) \phi_n = -\lambda_n \phi_n^\dagger \gamma_5 f(aD) \phi_n \quad (43)$$

where we applied the property:

$$\phi_n^\dagger \gamma_5 D = \lambda_n \phi_n^\dagger. \quad (44)$$

As $\lambda_n \neq 0$, then $\phi_n^\dagger \gamma_5 f(aD) \phi_n = 0$. Thus we immediately deduce that

$$\sum_{\lambda_n \neq 0} \phi_n^\dagger \gamma_5 f(aD) \phi_n = 0. \quad (45)$$

Inserting this result back into (42), we obtain a result which is identical to the one in [9], namely

$$\sum_n \phi_n^\dagger \gamma_5 f(aD) \phi_n = \sum_{\lambda_n=0} \phi_n^\dagger \gamma_5 \phi_n = n_+ - n_- = \text{index}(D). \quad (46)$$

The above theorem, (45) and (46), means that

$$\text{Tr} \Gamma_5 = n_+ - n_- , \quad (47)$$

where $\Gamma_5 = \gamma_5 f(aD)$. Because of (10), this operator anticommutes with the Hermitian operator $\mathcal{D} = \gamma_5 D$:

$$\mathcal{D} \Gamma_5 + \Gamma_5 \mathcal{D} = 0 , \quad (48)$$

which is identical to the relation found by Fujikawa in [10].

As for the interpretation of the $\text{Tr}\gamma_5 = 0$ relation, discussed in [10] for the case of the GWR, we follow a parallel approach. We should therefore find the ‘highest states’, namely, zero modes of Γ_5 . These are again responsible for the cancellation of the contribution coming from the zero modes of D to the trace of γ_5 . Denoting by ϕ those highest states, we should then have

$$\gamma_5 f(aD)\phi = 0 , \quad (49)$$

but this is precisely the relation we found in (25), namely, those vectors do exist *as a consequence of the good large momentum behaviour of D* . As for their number (degeneracy), it should be the same as for the zero modes of \mathcal{D} , since the degeneracy depends on the symmetries (and they are the same for both operators).

4 Examples

In ’t Hooft’s approach [11], the fermionic determinant is defined *in the continuum* by

$$\det D = \prod_i [\det(D_c + \Lambda_i)]^{e_i} \quad (50)$$

where

$$D_c = \not{D} = \gamma_\mu D_\mu , \quad D_\mu = \partial_\mu + iA_\mu . \quad (51)$$

Here D_c stands for ‘continuum’ Dirac operator, and indeed it also verifies the chirality condition (2) satisfied by the *lattice* operator D_c . To render $\det D$ finite (non-perturbatively), the gauge field A_μ is defined through an extrapolation mechanism from the usual link variables [11], and besides, Pauli-Villars regulators are introduced in (50), verifying the usual conditions:

$$\sum_i e_i \log \Lambda_i \equiv \log \Lambda , \quad \sum_i e_i \Lambda^n \log \Lambda_i \equiv 0 . \quad (52)$$

Here n runs from 0 through 4 if one is dealing with a four-dimensional theory. Of course, this range will be different for different numbers of dimensions. In 4 dimensions, three regulator fields are sufficient, while in 2 dimensions only one. We shall now see that D will satisfy a GWR when there is only one regulator, and a GGWR when three (or more) regulators are required.

The simplest case corresponds to using just one regulator field, which, in order to improve the UV behaviour of the bare Dirac operator has to have bosonic statistics. This is frequently stated in terms of a regulated action S_F^R

$$S_F^R = \int d^d x \left[\bar{\psi}(x) \not{D} \psi(x) + \bar{\phi}(x) (\not{D} + \Lambda) \phi(x) \right] \quad (53)$$

where ϕ is the bosonic regulator field ⁴. Equivalently, if one integrates out the regulator, the result is the equivalent action (for which we use the same notation)

$$S_F^R = \int d^d x \bar{\psi}(x) D_R \psi(x) \quad (54)$$

where

$$D_R = D_c \left(\frac{1}{\Lambda} D_c + 1 \right)^{-1}. \quad (55)$$

This is not the usual way of writing the action, since it is formally non-local, and the version (53) is sometimes preferred. However, (54) is the more convenient form to study the GWR: the operator D_R , as it is evident by comparing (55) with (6), will verify the relation

$$D_R \gamma_5 + \gamma_5 D_R = \frac{2}{\Lambda} D_R \gamma_5 D_R \quad (56)$$

i.e., a GWR with $a = \frac{2}{\Lambda}$. We can of course borrow some of the already known properties of the GWR to this case. For example, the regularized action (54) is invariant under the transformation:

$$\begin{aligned} \delta \psi &= i \xi \gamma_5 \left(1 - \frac{D_R}{\Lambda} \right) \psi = i \xi \gamma_5 \frac{\Lambda}{D_c + \Lambda} \psi \\ \delta \bar{\psi} &= i \xi \bar{\psi} \left(1 - \frac{D_R}{\Lambda} \right) \gamma_5 = i \xi \bar{\psi} \frac{\Lambda}{D_c + \Lambda} \gamma_5 \end{aligned} \quad (57)$$

where we have expressed the transformations both in terms of the regularized and unregularized Dirac operators.

This symmetry for the continuum theory has consequences for the study of anomalous Ward identities, from the point of view of anomalous (Fujikawa) Jacobians. Let us recall that, in the usual setting [12], one starts

⁴This regularized action was also introduced in the GWR context by Fujikawa [10], although not in order to show that it satisfies the GWR.

from the functional integral representation for the unregularized fermionic determinant D_c ,

$$\det D_c = \int \mathcal{D}\bar{\psi} \mathcal{D}\psi \exp\{-S_F\} \quad (58)$$

and then performs a local axial transformation for the fermions:

$$\begin{aligned} \delta\psi(x) &= i\xi(x) \gamma_5 \psi(x) \\ \delta\bar{\psi}(x) &= i\xi(x) \bar{\psi}(x) \gamma_5 . \end{aligned} \quad (59)$$

This produces a variation in the action:

$$\delta S_F = -i \int dx \xi(x) \partial_\mu J_{\mu 5} \quad (60)$$

where $J_{\mu 5} = \bar{\psi} \gamma_\mu \gamma_5 \psi$, and the measure acquires an anomalous Jacobian J :

$$\mathcal{D}\bar{\psi} \mathcal{D}\psi \rightarrow J \mathcal{D}\bar{\psi} \mathcal{D}\psi . \quad (61)$$

J is ill defined,

$$J = \exp[-2i\mathcal{A}] , \quad \mathcal{A} = \text{Tr}(\xi \gamma_5) . \quad (62)$$

A regulator ρ is then introduced to give meaning to the trace,

$$\mathcal{A} = \int dx \xi(x) \text{tr}[\gamma_5 \rho(\frac{\not{D}^2}{\Lambda^2})] \quad (63)$$

where $\rho(0) = 1$, and ρ vanishes rapidly for $x \rightarrow \infty$. Evaluating the regulated Jacobian and taking then the $\Lambda \rightarrow \infty$ limit yields the well known result:

$$\mathcal{A} = \int dx \xi(x) \frac{1}{32\pi^2} \epsilon^{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta} \quad (64)$$

which leads to the anomalous Ward identity

$$\partial_\mu \langle J_{\mu 5} \rangle = \frac{1}{16\pi^2} \epsilon^{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta} . \quad (65)$$

On the other hand, as we know about the existence of the GWR and its symmetry for the continuum theory, can start from the regularized action (53), perform the transformation (57), and then evaluate the anomalous Jacobian, *which does not need any extra regularization*, since its exponent is proportional to the functional trace:

$$\text{Tr} \left[\gamma_5 \left(1 - \frac{D_R}{\Lambda} \right) \right] \quad (66)$$

and moreover, in terms of $D_c = \not{D}$,

$$\text{Tr} \left[\gamma_5 \left(1 - \frac{D_R}{\Lambda} \right) \right] = \text{Tr} \left[\gamma_5 \frac{1}{1 + \frac{\not{D}}{\Lambda}} \right] = \text{Tr} \left[\gamma_5 \frac{1}{1 + \frac{\not{D}^2}{\Lambda^2}} \right]. \quad (67)$$

And (67) is one of the general expressions which lead to the proper result for the chiral anomaly in the continuum, when $\Lambda \rightarrow \infty$. The variation of the action, just produces the divergence of a regularized current. We thus again obtain the usual result, but note that the procedure has the advantage of involving regular objects at all the steps in the calculation.

It is well known that more general Pauli-Villars regularizations amount to replacing the simple equation (55) by:

$$\frac{D_R}{\Lambda} = \mathcal{R} \left(\frac{D_c}{\Lambda} \right), \quad (68)$$

where \mathcal{R} denotes a rational function, and Λ is the cutoff. The function \mathcal{R} will depend also on a set of dimensionless constants.

It is obvious that (68) plays a similar role to (18), and in order to find the corresponding f (and hence the GGWR), we have to find the solution to

$$y = \frac{x}{f(x)}. \quad (69)$$

For the particular case of a four dimensional theory, and if one wants to have convergence for all the diagrams contributing to the fermionic determinant (vacuum diagrams factorized) the rational function is

$$\mathcal{R}(x) = y \frac{y + \sqrt{2}}{(y + 1)^2}. \quad (70)$$

This expression was obtained by integrating the regulator fields in the corresponding regularized action:

$$S_F^R = \int d^4x \sum_{s=0}^3 \bar{\psi}_s (D_c + M_s) \psi_s \quad (71)$$

where ψ_0 and ψ_1 are fermionic fields, while $\psi_{2,3}$ are bosonic. The masses are: $M_0 = 0$, $M_1^2 = 2\Lambda^2$, $M_2^2 = M_3^2 = \Lambda^2$.

For this particular case, the relation can, indeed, be inverted to obtain y as a function of x , and so we can find the form of the function $f(x)$, which determines the GGWR and its associated global symmetry. The answer is:

$$f(x) = \frac{\sqrt{2} (1-x) x}{-1 + \sqrt{2} x + \sqrt{1 - 2(\sqrt{2} - 1) x}} \quad (72)$$

We have seen that the continuum Dirac operator plays here the same role as the lattice operator D_c , whose drawback is its non-locality. This is also true of 't Hooft's method, since the continuum Dirac operator, when viewed from the lattice point of view, is a non-local object.

As another example of a Dirac operator which verifies a GGWR, we point our Slavnov's proposal [13]. It amounts to using, on the lattice, a SLAC derivative lattice Dirac operator (which has no doublers and is non-local), supplemented by Pauli-Villars regulator fields. These regulators, with cutoffs of the order of a^{-1} , have the effect of taming the lattice artifacts introduced by the SLAC derivative operator. Indeed, it has been shown that more than one regulator field may be necessary to have a reasonable behaviour, even in two dimensions [13]. When three regulators are used, we are in the situation of the previously discussed example, where a GGWR relation is verified.

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